CHAPTER 13: PORTFOLIO CHOICE

by Jon D Kendall (kendall@pacific.net.sg)

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We round off the discussion of financial markets by considering the individual's selection of an appropriate **portfolio** of financial assets. The portfolio selection problem maybe summarised as follows: given that the investor has a limited stock of wealth, how should he or she distribute this wealth across the wide range of available financial assets? Suppose, for example, you inherited one million dollars which you wanted to invest. Even if you were interested only in investing in Australian stocks and bonds, there are still many different possible combinations of these assets which would total up to one million dollars. Which combination is the best?

Modern portfolio theory maintains that there are two fundamental factors governing this decision: the mean **return** and the **risk** on the return which the individual can expect from the entire portfolio of investments. So, logically, the portfolio problem can be divided into two parts. The first part involves the calculation of the combinations of assets which makes mean returns from the portfolio as large as possible, for given levels of risk. This calculation yields a set of **efficient portfolios.** The second part of the problem involves choosing the most preferred member of the set of efficient portfolios.

The first problem is solved through **portfolio selection theory** (developed by Harry Markowitz in the 1950s). Portfolio selection theory can be used to select a group or portfolio of assets that will yield higher returns than any other portfolio, consistent with a given level of risk. We shall see that the overall risks associated with a portfolio will in general be reduced by including more assets in the portfolio. In other words, there may be gains from diversification of the portfolio. Intuitively, investing in more than one asset reduces the risk of losing everything if the investment turns sour. But it is not just the number of assets invested in which reduces this possibility. Markowitz discovered that the intereaction *between* the assets comprising the portfolio held the key to reducing variability of returns.

With regard to the second problem, the **Expected Utility Maximisation** hypothesis of Von Neumann and Tobin can provide a solution. Section 15.1 presents a discussion of statistical concepts such as expected return, variance, standard deviation, covariance, and correlation. These concepts are then utilised in section 15.2 where we find a discussion of portfolio selection. Section 15.3 works through how investors choose their preferred portfolio from the feasible set. Finally section 15.4 generalises the model and also looks at limitations of the portfolio approach.

15.1: STATISTICAL CONCEPTS[1]

15.1.1: The Role of Probability in Portfolio Analysis

Portfolio analysis is premised on a world of uncertainty. If we knew for certain that a particular stock will pay higher dividends than any other stock or bond, we would not hesitate to invest exclusively it that asset alone. The fact is, that we do not have certain information about the future return and thus, must rely on the past for its prediction. Asset return is thus modelled as a **discrete random variable** which can be described in terms of a **probability distribution**. A probability distribution is a listing of every possible outcome the random variable can take on, multiplied by its respective probability of occurrence.

15.1.2: Mathematical Expectation

An associated concept is **mathematical expectation**. We may define expectation as:

$$E(X) = x_1\pi_1 + x_2\pi_2 + \ldots + x_n\pi_n = \Sigma_i x_i \pi_i \qquad (15.1)$$

[1] Further discussion of portfolio analysis requires a knowledge of basic statistics. Whilst a basic outline is given here, you are encouraged to obtain more detailed information from an introductory statistics textbook such as Mendenhall, Reinmuth, and Beaver (1989). Alternatively, Markowitz (1991) contains an excellent development of basic statistics as well as the definitive discussion of portfolio theory.

where π_i is the probability of event i with the properties that

$$0 \le \pi_i \le 1 \qquad \text{and} \qquad \pi_1 + \pi_2 + \ldots + \pi_n = 1. \tag{15.2}$$

The expected value is the weighted average of the random variable. We define the "mean" as the mathematical expectation:

$$\mu_x = E(X) \tag{15.3}$$

We define the variance (in equivalent forms) as:

$$\sigma^2_x \equiv V(X) = \Sigma_i(X_i - \mu_x)^2\pi_i = E[(X - \mu_x)(X - \mu_x)] = E[(X - \mu_x)^2] \tag{15.4}$$

Variance is a measure of the dispersion or variability around the expected value. Sometimes it is useful to express the measure of dispersion in the same units as the original data. Standard deviation, which is the square root of the variance, is used for this purpose.

Several rules for manipulation of expectations operators are useful. For expectation we have:

$$E(cX) = cE(X) \tag{15.5}$$

where c is a constant

$$E[g(X) + h(x)] = E[g(X)] + E[h(X)] \tag{15.6}$$

where g(X) and h(X) are functions of X.

$$E(c + X) = E(c) + E(X) \tag{15.7}$$

$$E(X + Y) = E(X) + E(Y) \tag{15.8}$$

For variance we have:

$$V(cX) = c^2V(X) \tag{15.9}$$

$$V(c + X) = V(X) \tag{15.10}$$

$$V(X + Y) = V(X) + V(Y) + 2Cov(X,Y) \tag{15.11}$$

where Cov(X,Y) is defined *anon*.

Example 15.1: Expected Value and Variance

An example will assist in understanding the concepts of expected value and variance. Suppose we have a portfolio consisting of ownership in two small companies. One company sells fashion sportswear whilst the other one sells bakery products. On the basis of past experience, annual returns for the two companies as well as the two probability distributions follow below:

Lim's Sportswear:

Historical Data

Year	**Return (%)**
1989	10
1990	100
1991	10
1992	-100

The return in each year has 1/4 probability of occurring. Since the 10% return occurs in two years, it has a total probability of 2/4. Each of the other possible returns occur only once so they each have 1/4 probability of occurring. The above information allows construction of a:

Probability Distribution

Return	**Probability**
-100	1/4
10	2/4
-100	1/4

Expected Return: - 100 (1/4) + 10 (2/4) + 100 (1/4) = 5 percent.

Variance: $(-100 - 5)^2$ (1/4) + $(10 - 5)^2$ (2/4) + $(100 - 5)^2$ (1/4) =
11025 (1/4) + 25 (2/4) + 9025 (1/4) = 5025

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Alternatively, we may calculate the variance

Return (%)	Probability	Deviation from the Average	Deviations, Squared	Probability Times Deviations, Squared
-100	1/4	(-100 - 5) = -105	11025	2756.25
0	2/4	(10 - 5) = 5	25	12.5
100	1/4	(100 - 5) = 95	9025	2256.25
Sum	**1**			**5025**

Choo's Bakery:

Historical Data

Year	Return (%)
1989	20
1990	5
1991	5
1992	-10

from which is constructed:

Probability Distribution

Return (%)	Probability
-10	1/4
5	2/4
20	1/4

-3-

Expected Return: $-10\ (1/32) + 5\ (30/32) + 20\ (1/32)$
$= 5$ percent.

Variance: $(-10 - 5)^2\ (1/4) + (5 - 5)^2\ (2/4) + (20 - 5)^2\ (1/4)$
$= 225(1/4) + 0(2/4) + 225(1/4) = 112.5$

The difference between the figures help to make clear the meaning of expected value and variance. Notice that, in this example, the expected return is the weighted average of returns and is the same for both businesses. Notice also that it is more variable for the sportswear shop. The return has a greater range of returns. It can range from -100 percent to +100 percent whereas the Bakery ranges only from -10 percent to +20 percent. This is reflected in the higher variance figure. Careful examination of the definition of variance shows it to be the sum of squared differences between the mean and the actual value taken on by the random variable, multiplied by the respective probability of occurrence. The larger are these differences or deviations, the larger will be the variance. They will also be larger the greater is the probability of high losses or gains.

15.1.3: Covariance and Linear Correlation

A generalisation of the variance measure is covariance, which allows us to measure the extent of dispersion between different random variables. That is, it measures how random variables vary together. We define covariance as:

$$\mathrm{Cov}(X,Y) = \Sigma_i\Sigma_j[(x_i - \mu_x)(y_j - \mu_y)\pi_{ij}] = E[(X - E(X))(Y - (E(Y))] \qquad (15.12)$$

where expectations work as before and π_{ij} is a **joint probability density function**. This function is defined as:

$$\pi_{ij} = \Pi(X = x_i, Y = y_j) \tag{15.13}$$

It has several properties, *viz.*

$$\pi_{ij} \geq 0 \quad \text{and } \Sigma_i\Sigma_j\pi_{ij} = 1 \tag{15.14}$$

The first expression means that each joint probability must be greater than or equal to zero. The second means that the sum of the probabilities equals one and implies that no probability may be greater than one.

We also define **marginal probabilities** as:

$$\pi_i = \pi_1 + \pi_2 + \ldots + \pi_m = \Sigma_j\pi_{ij}$$

and

$$\pi_j = \pi_1 + \pi_2 + \ldots + \pi_n = \Sigma_i\pi_{ij} \tag{15.15}$$

Instead of considering what is the probability of occurrence for each possible value taken on by a single random variable, we must consider the probabilities of all possible joint occurrences of values taken on by two random variables. Obviously a special case of covariance is variance where the two random variables are the same. In such a situation, the formula for covariance is the same as the formula for variance. However, unlike the case for variance, covariance can be either negative or positive, depending on how the two random variables vary together. If they tend to move in the same direction (i.e. both variables take values above and below the mean at the same time), covariance will be positive. If they tend to move in opposite directions, covariance will be negative (i.e. one variable takes on values above the mean at the same time the other variable takes on values below the mean). Finally, if they tend to offset one another, covariance will be close to zero.

Additional insight into the meaning of covariance may be gleaned by examining a related concept known as **linear correlation.** Linear correlation is covariance standardised to a number which can vary between -1 and 1. It is defined as:

$$r = \text{cov}(X,Y)/\sigma_X \, \sigma_Y \qquad (15.16)$$

Again, notice the special case where X and Y are the same variable. In that case Cov(X,Y) will equal the product of σ_X and σ_Y and r will thus equal one.

Example 15.2 Covariance and Linear Correlation

We may continue with our two-asset portfolio model to illustrate the concepts of covariance and linear correlation. Suppose that the joint probability density function (derived from the historical data above) for Lim's Sportswear and Choo's Bakery is as follows:

Choo's Bakery (Y) (%)

Lim's Sportswear (X) (%)	-10	5	20	$g(y_j)$
-100	1/4	0	0	1/4
10	0	1/4	1/4	2/4
100	0	1/4	0	1/4
$f(x_i)$	1/4	2/4	1/4	4/4 = 1

Each cell tells us the joint probability of returns for the two businesses. For example, there is a 1/4th probability that both businesses will yield negative returns. $f(x_i)$ and $g(y_j)$ are known as **marginal density functions**. They are the density functions stated earlier for the respective businesses.

We already know:

E(X) = E(Y) = 5% and
$V(X) = \sigma^2{}_X = 5025; \sigma_X = 70.89$
$V(Y) = 112.5 = \sigma^2{}_Y; \sigma_Y = 10.61$

Covariance may be calculated as:

Cov(X,Y) = (-100 - 5) (-10-5)(1/4) + (-100-5)(5-5)(0) + (-10005)(20-5)(0)

+ (10-5)(-10-5)(0) + (10-5)(5-5)(1/4) + (10-5)(20-5)(1/4)

+ (100-5)(-10-5)(0) + (100-5)(5-5)(1/4) + (100-5)(20-5)(0)

= 1575(1/4) + 0 + 0 + 0 + 0 + 75(1/4) + 0 + 0 + 0

= 412.5

correlation = r = (412.5/ (70.8910.61)) = 412.5/752.14 = 0.548

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It follows that covariance can be retrieved from correlation by noting that:

$$\mathrm{Cov}\,(X,Y) = \sigma_X\,\sigma_Y\,r \qquad (15.17)$$

So,

$$\mathrm{Cov}\,(X,Y) = (70.89)(10.61(0.548) = 412.5$$

Notice that there is some linear correlation between the sportswear business and the bakery. That is, the two returns vary together positively. If r had equalled one, there would have been perfect correlation.

15.1.4: Summing Random Variables

What would be the expected value and variance of the return of the two businesses above summed together? We know from equation 15.8 that the sum of the expected value would be

$$E(X + Y) = E(X) + E(Y) = 5 + 5 = 10$$

We defined the sum of the variance of two variables in equation 15.11 as:

$$V(X + Y) = V(X) + V(Y) + 2\,\mathrm{Cov}(X,Y)$$

Thus, we have:

$$V(X + Y) = 5025 + 112.5 + (2) \times 412.5 = 5962.5$$

Notice that the sum of the variances is more than the simple sum. Account must be taken of the covariance between the two variables. The only instance in which the sum of the variances equals the simple sum is when there is no covariation between the variables. This can occur if the two variables are "independent." That is, if the occurrence of one variable does not alter the probability distribution of the second variable.

The concepts of expectation, variance, standard deviation, covariance, and correlation, equip us with the basic tools to tackle the two components of the portfolio problem, namely deriving the **portfolio opportunity set** and allowing the investor to select a preferred portfolio.

15.2: THE PORTFOLIO PROBLEM

15.2.1: The Key Assumptions

Selection of a preferred portfolio by an investor relies on several key assumptions. Firstly we assume that, *ceteris paribus***,** investors prefer a higher expected return to a lower expected return. Clearly this is not unreasonable. Secondly we assume investors prefer lower fluctuations in returns to higher fluctuations. That is, they prefer lower variance of expected returns to higher variance, where variance is synonymous with risk. We are thus assuming investors to be **risk averse**. Implicitly, we are assuming that expected return and variance are all that are necessary for an investor to characterise a portfolio. Together these assumptions imply the existence of a positive relationship between expected return and variance; the investor will accept a higher variance only for a higher rate of return.

But, before we look at investor preferences, we must consider what **choices** are available to them in terms of the risk and return *produced* by different portfolios.

15.2.2: Portfolio Diversification

We now need to examine broad principles arising out of the statistics derived above. One such principle is a consequence of summing random variables. If we add assets to a portfolio, we reduce its overall average variability. In the first instance we will assume there to be no covariance between assets in the portfolio so as not to complicate the issue. Later we shall see the crucial role played by covariance or correlation between assets. The example below illustrates how overall variability of the portfolio can be reduced just through adding assets to it. It then shows how covariance can enhance diversification.

Example 15.3: Portfolio Diversification

Suppose expected returns and variances for three companies, listed below, are considered.

Company	**Expected Return (%)**	**Variance**
X	5	100
Y	5	100
Z	5	100

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Also Cov(X,Y) = Cov (X,Z) = Cov (Y,Z) = 0

Obviously the expected returns and variances for the companies are identical. Suppose we pool together the assets equally to form a portfolio. For comparability we are interested in the average expected values and average variances of the different possible combinations.

$E_{port\ (X)} = 5$

$E_{port\ (X,Y)} = (5 + 5)/2 = 5$

$E_{port\ (X,Y,Z)} = (5 + 5 + 5)/3 = 5$

$V_{port(X)} = 100$

$V_{port\ (X,Y)} = V((X + Y)/n) = V(X + Y)/n^2 = V(X) + V(Y) +$

$2Cov(X,Y)/n^2 = (100 + 100 + 2^*0)/2^2 = 50$

$V_{port\ (X,Y,Z)} = (100 + 100 + 100)/32 = 33.33$

Expectations remains at 5 percent for all portfolios. Nevertheless, variance is **reduced** by pooling the assets together. It falls from 100 to 50 to 33.33. This is the benefit of portfolio diversification. The more assets contained within the portfolio, the lower the variance. Of course we made a very important assumption about covariance or correlation, **viz**, that they equalled zero. In general this is not the case. However, the impact of covariance is easy to see by recalling the formula for summing random variables from equation 15.11. That is,

$V(X + Y) = V(X) + V(Y) + 2Cov(X,Y)$

or substituting in equation 15.17 and revising notation slightly we obtain,

$V(X + Y) = \sigma^2_X + \sigma^2_Y + 2\sigma_X \sigma_Y r$

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With zero correlation the variances of X and Y were just added together. But, suppose that the correlation was positive, say with a coefficient of 1. In that case the portfolio variances for just X and X and Y would appear as:

$V_{port\ (X)} = 100$

$V_{port\ (X,Y)} = V((X + Y)/n) = V(X + Y)/n^2 = (V(X) + V(Y) + 2Cov\ (X,Y))/n^2 =$
$(100 + 100 + 2^* 10^* 10^* 1)/2^2 = 100$

Notice that there is no change in variance. Adding Corporation Y does not reduce variance. It remains at 100.

Now suppose that correlation between corporations X and Y is equal to -1. Portfolio variances for X and X with Y would appear as:

$V_{port(X)} = 100$

$V_{port(X,Y)} = V((X + Y)/n) = V\ (X + Y)/n^2 = (V(X) + V(Y) + 2Cov(X,Y))/n^2 =$
$(100 + 100 - 2\times 10\times 10\times 1)/2^2 = 0$

In this case portfolio variance fell to zero with the inclusion of Corporation Y because variance for Corporation Y was exactly opposite that of Corporation X.

On the basis of the above example we may generalise about the impact of correlation on the benefits of portfolio diversification. The usefulness of adding assets depends on the extent to which the additional asset is correlated with existing assets. Adding assets to a portfolio will reduce variance so long as correlation between the new asset and each of the assets within the portfolio is zero or negative. Assets which are perfectly negatively correlated (-1) will offset each other to the extent that variance can be eliminated altogether.

15.2.3: Deriving the Portfolio Opportunity Set

We need to construct a **portfolio opportunity set** in order to identify risk and return choices available to investors. From this set we are interested in the best combinations of assets in terms of risk and return. This involves maximising expected return, subject to a given amount of variance. That is, we want to combine different quantities of assets such that we maximise the overall return, subject to variance being at or below a certain level. Alternatively the problem can be looked at as one of minimising variance subject to a given expected rate of return. The point is to derive the locus of **efficient** portfolios from the set of *all* possible combinations of assets. What we mean by an efficient combination is that it is impossible to obtain a smaller variance without reducing expected rate of return for the given set of assets included in the portfolio (Markowitz 1991, p. 22). Obviously an investor who is averse to risk as measured by variability only will be interested in portfolios that lie along the efficient frontier.

Assume a portfolio holder invests her money such that it is split between two assets. Portfolio return may be defined as:

$$E_{port} = (s_X)E(X) + (1 - s_X)E(Y) \qquad (15.18)$$

We may define the variance of such a portfolio as:

$$V_{port}^2 = (s_X)^2V(X) + (1 - s_X)^2V(Y) + s_X(1 - s_X)Cov(X,Y) + (1-s_x)s_XCov(Y,X)$$

$$= (s_X)^2V(X) + (1 - s_X)^2V(Y) + 2s_X(1 - s_X)Cov(X,Y) \qquad (15.19)$$

where s_X and 1 - s_X represent a **convex combination**. A convex combination can be thought of as a weighted average of the shares of every asset dollar invested. We allow the weights to vary, allowing a plethora of possible combinations of assets. The different possible portfolios, together, form the portfolio opportunity set. However, we are interested only in the efficient set.

15.2.4: The Optimal Portfolio with Minimum Variance

It is possible to derive a particular point on the portfolio opportunity set. This point gives the combination of the two assets with the least overall variance. Through calculus (see Elton & Gruber, 1991, p. 46) it can be shown that the variance-minimising proportion (s_X) is:

$$s_X = (V(Y) + Cov(X,Y)) / (V(X) + V(Y) + 2Cov(X,Y))$$

The following example shows how to derive the variance-minimising proportion (s_x) as well as how to graph the efficient set of portfolios.[2]

[2] We do not actually derive the efficient set of points as that goes beyond the scope of this chapter. The solution involves a non-linear programming problem. See Eck (1976, Chapter 19).

Example 15.4: The Efficient Portfolios

We are interested in the most efficient combinations of the two assets as discussed above. That is, we are interested in finding the combination which ensures minimisation of the portfolio variance for given levels of rates of return. Suppose that we had a choice of buying some combination of shares in two corporations. Historical data for Wytec International, a clothes manufacturer and the Dilladong Company, a manufacturer of boomerangs are as follows:

Wytec International (X)

Historical Data

Year	**Return (%)**
1989	5
1990	3
1991	1
1992	2

Dilladong Corporation (Y)

Historical Data

Year	**Return (%)**
1989	-3
1990	15
1991	12
1992	5

Summary Information

Corporation	**Expected Return (%)**	**Variance**
Wytec Int'l (X)	2.75	2.19
Dilladong Co (Y)	7.25	48.19
Cov (X,Y) = -6.94 Correlation(r) = -0.68		

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The table below displays expected returns and variances for different possible combinations of the two corporations which could make up a portfolio.

S_X	0.0	0.1	0.3	0.5	0.7	0.9	1.0
E_{port}	7.25	6.80	5.90	5.00	4.10	3.20	2.75
V_{port}	48.19	37.81	20.90	9.13	2.5	1.01	2.19

Notice that with s_x = 0.7 and 0.9 variances are lower *and* returns are higher than for the portfolio consisting only of Wytec Corporation. The negative correlation between returns of Wytec and Dilladong assists in dampening the portfolio variance.

The solution to finding efficient portfolios was solved by Markowitz in the 1950s. The key was taking account of covariances between asset returns. In order to understand exactly what is the difference between efficient and inefficient portfolios we begin by determining the weighted combination of assets which *minimises* variance of the portfolio for a given rate of return.

For the two corporations, the variance-minimising proportion is:

$s_X = (48.19 - (-6.94)) / (2.19 + 48.19 - 2(-6.94))$
$= (55.13 / 64.26) = 0.86$ so, $1 - s_X = 0.14$.

For every investment dollar, 86 cents should be invested in Wytec whilst 14 cents should be invested in Dilladong if we wish to minimise portfolio variance. Overall
return and variance of the portfolio would then be:

$E_{port} = (0.86) \times 2.75 + (1\text{-}0.86)\times 7.25 = 3.38\%$

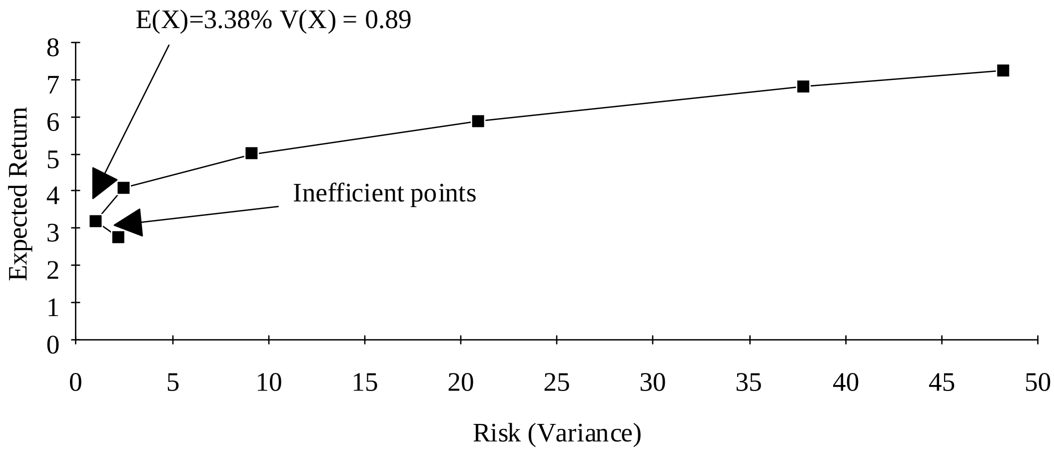
$V_{port} = (0.86)^2 \times (2.19) + (1 - 0.86)^2 \times (48.19) + 2\times 0.86\,(1\text{-}0.86)\times(-6.94)$

$= 0.89$

Notice again how adding the riskier asset (Dilladong Company) not only *reduces* risk but also yields a return *greater* than that of Wytec International alone.

More generally, we can examine visually the efficient and inefficient combinations of the two assets with regard to risk (proxies by variance) and return. We can measure risk on the horizontal axis and expected return on the vertical axis on the chart below.

Figure 15.2.1

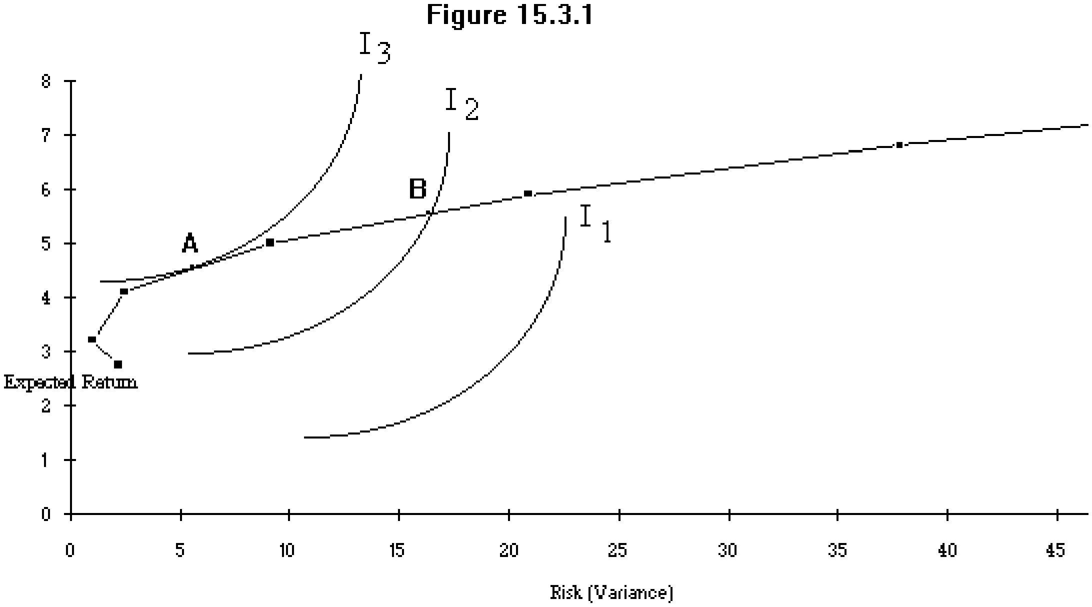


Notice that the minimum risk point corresponds to a variance of 0.89 consistent with a return of 3.38%. We can understand the meaning of inefficient points better by looking at the decision of buying Wytec only or some combination with Dilladong consistent with variance of 2.19. As it happens if we buy only Wytec we have an expected return of 2.75. But, by noting the efficient combination to the right of this point, we note that the combination of s_x = 0.716 of Wytec and (1 - s_x) = 0.284 of Dilladong yields us the *same* variance of 2.19 but a *higher* return of 4%. Notice also that points along the line higher than the minimum variance imply a positive trade off between risk and return. Higher return can be obtained only at the cost of higher risk.

15.3: EFFICIENT PORTFOLIOS AND INVESTOR PREFERENCES

The positive trade off between risk and return begs the question of just which efficient combination to choose along the curve in Figure 15.2.1. At this stage it is clear that only points along the efficient frontier would be chosen as they yield the lowest risk for given rates of return. Combinations within the frontier are inferior. However, one investor may prefer to undertake higher risk in order to obtain a higher return. Another may not wish to tolerate such high risk and opt for a lower return. Such choices are decided by the preferences of the individual investor. The one which maximises her expected utility of wealth will be chosen. We have made the implicit assumption that investors are, at least to some degree, risk averse. If so, then investors will choose combinations somewhere along the efficient frontier in accordance with individual preferences. We can show graphically this choice through introducing the notion of indifference curves.[3]

Figure 15.3.1



[3] The consumer theory underlying indifference curves is beyond the scope of this chapter. Additional information may be found in any intermediate microeconomics textbook.

An indifference curve represents the combinations of risk and return which yield the same utility, or satisfaction, to the investor. They will be bowed out reflecting risk aversion by the investor. But, the precise nature of the bowed shape is determined by the degree of risk aversion. Steeper indifference curves reflect a higher degree of risk aversion. Note also that the higher the indifference curve is in the graph, the more utility received by the investor. Point A, along indifference curve I_3, represents the preferred portfolio chosen by the investor. Even though Point B is on the efficient frontier, it is *not* chosen because it is dominated by point A, which is on a higher indifference curve.

15.4 EXTENSIONS AND QUALIFICATIONS

15.4.1: Generalising Portfolio Analysis to More than Two Assets

Thus far we have discussed portfolio selection with only two assets. The principles discussed above apply equally to the case with more than two assets. The only modification is in the portfolio formulae to allow for inclusion of more assets. Suppose we have 3 assets to include in a portfolio. Portfolio return may be defined as:

$$E_{port} = (s_1)E(1) + (s_2)E(2) + s_3E(3) \qquad (15.20)$$

where $s_1 + s_2 + s_3 = 1$.

Calculation of the portfolio variance becomes exceedingly complex as more assets are added. The problem stems from having to calculate every possible pair of covariances. We may employ a trick to remember what is included in the formula.[4] We employ the use of a covariance matrix as shown below for the case of three assets:

Asset	**1**	**2**	**3**
1	Cov (1,1)	Cov (1,2)	Cov (1,3)
2	Cov (2,1)	Cov (2,2)	Cov (2,3)
3	Cov (3,1)	Cov (3,2)	Cov (3,3)

[4] See Eck (1976, Chapter 19).

We need only recall that Cov(1,1) is the variance of asset 1. It is also the case that Cov(1,2) = Cov(2,1). Similar arguments apply to the remaining asset combinations. The variance of the portfolio is then just the sum of all elements of the matrix, multiplied by the respective share. Thus we have:

$$V_{port} = (s_1)^2 Var(1) + s_1 s_2 Cov(1,2) + s_1 s_3 Cov(1,3) + s_2 s_1 Cov(2,1) + s_2 2Var(2) + s_2 s_3 Cov(2,3) + s_3 s_1 Cov(3,1) + s_3 s_2 Cov(3,2) + s_3 2Var(3) \qquad (15.21)$$

Regardless of the number of assets to be included in the portfolio it is easy to remember the formula for variance in this fashion.

15.4.2: The Limitations of Portfolio Analysis

The assumption that expected return and variance are all that are necessary for an investor to characterise a portfolio should be looked at more closely. This assumption causes some difficulty as it implies that variance is a reasonable measure of risk. This may not be so as we must really consider the investor's individual preference structure in addition to the specific characteristics of the portfolio.

An example will make the problem clear. Suppose that one was given the choice between two stocks with the following discrete probability density functions:

Stock 1	Probability	Stock 2	Probability
Return (%)		**Return (%)**	
-5	1/4	-5	2/4
0	1/4	0	0
5	1/4	5	0
10	1/4	10	2/4

E(Stock 1) = E(Stock 2) = 2.5

V(Stock 1) = V(Stock 2) = 31.25.

Notice that the two stocks have identical means and variances; the investor should be indifferent between them. However, we need to look at the **expected utility** of each stock in order to know which one maximises satisfaction of the investor. It is possible that the investor's preferences are such that s/he prefers a uniform probability of returns (stock 1) to one with big gains or losses (stock 2). This difference is not picked up by mean and variance alone.

There are two ways out of this dilemma. The first is to suppose that the investor possesses a **quadratic utility function** so that she is indifferent between the two stocks. The other way is to suppose that investment returns are **normally distributed** with mean μ and variance σ^2. In this case the problem above is assumed away because normal distributions are characterised exclusively by mean and variance. However, if we are uncomfortable with either of these assumptions, then we must be cautious about inferring too much about how investors pick portfolios based on this method. We should also be cautious about assuming that variance is a perfect proxy for risk.

15.5: SUMMARY OF MAIN POINTS

This chapter began with a discussion of statistical concepts such as expectation, variance, covariance, and linear correlation. We then applied these statistical tools to the portfolio problem. We found that portfolio diversification reduced variance (which we used as a proxy for risk) when individual assets were uncorrelated. Accounting for covariance or linear correlation between assets allowed for another potential for reduction in risk. So long as assets were negatively correlated, their inclusion reduced risk. We then found the minimum-variance return of a portfolio consisting of two assets and then mapped out the efficient combinations yielding the lowest risk for given levels of expected return. Indifference curves were introduced to show how investors with differing degrees of risk aversion would select a preferred portfolio from the efficient set. Finally we generalised the portfolio model to take account of more than two assets. We also looked at limitations of portfolio analysis.

QUESTIONS FOR CHAPTER 15

1. Distinguish between variance and covariance.

2. Assume the following historical data:

Year	Asset 1 Return %	Asset 2 Return %
1990	5	20
1991	3	-10
1992	4	5
1993	6	9

a. Calculate the expected returns, variances, covariance, and linear correlation for the two assets.

Which one has the higher return?

Which one has the lowest variance?

b. Calculate the minimum variance portfolio consisting of the two assets.

Is it equal to the asset with the lessor variance?

Why or why not?

c. Sketch the efficient frontier of risk and expected return. Indicate inefficient points.

Why are they considered inefficient?

3. Use the example in the chapter dealing with Wytec and Dilladong to answer the following questions.

   a. Sketch the efficient frontier assuming perfect positive correlation (ie. $r = 1$) between the returns of the corporations. Now do the same assuming a perfect negative correlation (i.e. $r = -1$).

      How do these sketches compare with the chart in the chapter?

      How can you explain the differences?

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